**432. Proposed by C. N. SCHMALL, New York City.**

There are $n$ straight lines in a plane, no two of which are parallel and no three of which are concurrent. Their points of intersection being joined show that the number of new lines drawn is $\frac{1}{8}n(n-1)(n-2)(n-3)$.

SOLUTION BY LAENAS G. WELD, Pullman, Illinois.

Each of the $n$ given lines intersects each of the other $(n-1)$ lines, determining (since each point is thus twice determined) points to the number of $\dfrac{n(n-1)}{2}$.

Let $P_{ik}$ be the point determined by the lines $l_i$ and $l_k$. From $P_{ik}$ there may be drawn $R\left(=\dfrac{n(n-1)}{2}-1\right)$ lines to the other points. But the line $l_i$ contains $(n-2)$ of these other points and the line $l_k$ the same number. Hence the given lines $l_i$ and $l_k$ account for $2(n-2)$ of the above $R$ lines and the number of new lines is

$$R-2(n-2)=\left(\frac{n(n-1)}{2}-1\right)-2(n-2).$$

Since there are $\dfrac{n(n-1)}{2}$ points such as $P_{ik}$ the total number of new lines (since each is thus counted twice) is

$$\frac{1}{2}\cdot\frac{n(n-1)}{2}\left\{\left(\frac{n(n-1)}{2}-1\right)-2(n-2)\right\},$$

or

$$\frac{1}{8}n(n-1)(n-2)(n-3).$$

Also solved by J. W. CLAWSON, HERBERT N. CARLETON, H. C. FEEMSTER, WALTER C. EELLS, PAUL CAPRON, FRANK R. MORRIS, ELBERT H. CLARKE, N. P. PANDYA, and HORACE OLSON.

GEOMETRY.

**461. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.**

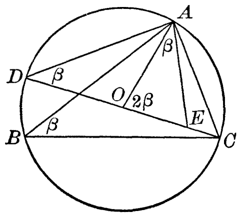
Prove by means of any inscribed triangle the following trigonometrical relations:

$\sin 2\beta = 2\sin\beta\cos\beta$; $\cos 2\beta=\cos^2\beta-\sin^2\beta$; $\sin 3\beta = 3\sin\beta-4\sin^3\beta$;

$\cos 3\beta = 4\cos^3\beta-3\cos\beta$.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let $ABC$ be the triangle. Join $A$ and $C$ to $O$, the center of the circumcircle. Extend $CO$ to meet the circle at $D$. Draw $AD$. Draw $AE$ making angle $OAE$ equal to angle $ABC$. Denote angle $ABC$ by $\beta$. Call the radius $R$.



(1) $$\sin 2\beta = \sin AOC = \sin ACD \cdot \frac{AC}{R} = \frac{AC \cdot AD}{2R^2} = 2\frac{AC}{2R} \cdot \frac{AD}{2R} = 2 \sin \beta \cos \beta.$$

(2) $$\cos 2\beta = \cos AOC = \frac{R^2 + R^2 - \overline{AC}^2}{2R \cdot R} = \frac{4R^2 - \overline{AC}^2 - \overline{AC}^2}{4R^2} = \frac{\overline{AD}^2 - \overline{AC}^2}{4R^2}$$
$$= \left(\frac{AD}{2R}\right)^2 - \left(\frac{AC}{2R}\right)^2 = \cos^2 \beta - \sin^2 \beta.$$

(3) $$\sin 3\beta \sin AEC = \sin AEO = \sin \beta \cdot \frac{R}{OE}.$$

Now
$$\frac{OE}{R} = \frac{AE}{AD} = \frac{AE}{2R \cos \beta}; \quad i.\ e.,\ AE = 2\overline{OE} \cos \beta.$$

Again
$$OE^2 = \overline{AE}^2 + R^2 - 2\overline{AE} \cdot R \cos \beta = 4\overline{OE}^2 \cos^2 \beta + R^2 - 4\overline{OE} \cdot R \cdot \cos^2 \beta;$$
$$\therefore \quad OE = \frac{R}{4 \cos^2 \beta - 1};$$
$$\therefore \quad \sin 3\beta = \sin \beta(4 \cos^2 \beta - 1) = 3 \sin \beta - 4 \sin^3 \beta.$$

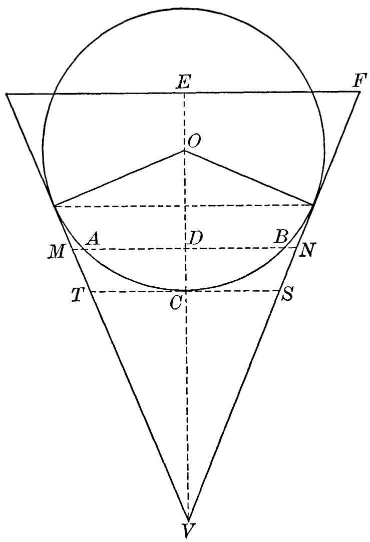
(4) $$\cos 3\beta = \frac{R^2 - \overline{AE}^2 - \overline{OE}^2}{2\overline{OE} \cdot AE} = \frac{(4 \cos^2 \beta - 1)^2 - 1 - 4 \cos^2 \beta}{4 \cos \beta} = 4 \cos^3 \beta - 3 \cos \beta.$$

**462. Proposed by DANIEL KRETH, Wellman, Iowa.**

A conical glass, the diameter of the base of which is 5 inches and altitude 6 inches, is one-fifth full of water. If a sphere 4 inches in diameter is dropped into it, how much of the vertical axis of the glass is immersed?

## SOLUTION BY J. A. CAPRON, Notre Dame University.

Let $MN$ be the level of the water after the sphere is dropped; $R = 5/2$ inches = radius of base of given cone; $r = 2$ inches = radius of sphere; $r_1$ = radius of base of cone $VMN$; $x$ = height of spherical segment $ACB$; $\alpha$ = semivertical angle of cone; and $CV = a$.




Then volume of cone $VMN = V_c = \frac{1}{3}\pi r_1^2(a+x)$; volume of segment $ACB = V_s = \pi x^2(r - x/3)$; and volume of given cone $= V = \frac{1}{3}\pi R^2 h$, where $h = 6$ inches.